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ABSTRACT ALGEBRA (Group)

1. Ques: The set of rational numbers is a group with respect to binary operation " addition".

Soln:- We know Q = Set of rational numbers

$$= \left\{ \frac{p}{q} : p, q \in I,\ q \neq 0 \right\}$$

(a) Closure law :- Sum of two rational numbers is again a rational number.

Example - $\frac{3}{4}, \frac{-7}{15} \in Q$

and their sum $= \frac{3}{4} + \left(\frac{-7}{15}\right) = \frac{3}{4} - \frac{7}{15} = \frac{45-28}{60} = \frac{17}{60} \in Q$

which is also a rational number.

(b) Associative law - Some element of Q are $\frac{2}{3}, \frac{5}{7}, -\frac{4}{9}$

Now $\frac{2}{3} + \left[\frac{5}{7} + \left(-\frac{4}{9}\right)\right]$ and $\left[\left(\frac{2}{3} + \frac{5}{7}\right) + \left(-\frac{4}{9}\right)\right]$ are equal $\left[\because \frac{2}{3} + \left\{\frac{5}{7} + \left(-\frac{4}{9}\right)\right\} = \frac{59}{63}\right.$ and $\left.\left[\left(\frac{2}{3} + \frac{5}{7}\right) + \left(-\frac{4}{9}\right)\right] = \frac{59}{63}\right]$

(c) Identity :- take $-\frac{7}{15} \in Q$

and $-\frac{7}{15} + 0 = 0 + \left(-\frac{7}{15}\right) = -\frac{7}{15}$

since $0 \in Q \Rightarrow 0$ is the identity element.

(d) Inverse :- Take $-\frac{17}{35} \in Q$

Now $-\frac{17}{35} + a = 0 = a + \left(-\frac{17}{35}\right) \Rightarrow a = \frac{17}{35}$ and $\frac{17}{35} \in Q$.

So, inverse axiom is satisfied.

Hence $(Q, +)$ is a group.

2. Que. The set of non-zero rational numbers is a group with respect to multiplication.

Soln:- let $Q_1 = Q - \{0\}$

(a) Closure :- Take $\frac{7}{3}, \frac{8}{15}$; $\frac{7}{3}, \frac{8}{15} \in Q_1$

Now, $\frac{7}{3} \times \frac{8}{15} = \frac{56}{45}$ and $\frac{56}{45} \in Q_1$

Take $\frac{1}{2}, -\frac{3}{7}$; $\frac{1}{2}, -\frac{3}{7} \in Q$

Now, $\frac{1}{2} \times -\frac{3}{7} = -\frac{3}{14}$ and $-\frac{3}{14} \in Q_1$

$\therefore$ closure law satisfied.

(b) Associative :- Take $\frac{1}{3}, \frac{2}{5}$ and $-\frac{3}{7} \in Q$

Now, we can easily verify that

$\frac{1}{3} \times \left[\frac{2}{5} \times \left(-\frac{3}{7}\right)\right] = \left(\frac{1}{3} \times \frac{2}{5}\right) \times \left(-\frac{3}{7}\right) = \frac{-2}{35}$

$\therefore$ Associative law satisfied.

(c) Identity :- 1 is the identity element and $1 \in Q$

Take any rational number $\frac{17}{38}$

Now $\frac{17}{38} \times 1 = \frac{17}{38} = 1 \times \frac{17}{38}$

$\therefore$ Identity exists.

(d) Inverse :- Take $\frac{2}{3} \in Q_1$

Now, $\frac{2}{3} \times a = 1 = a \times \frac{2}{3}$

$\Rightarrow a = \frac{3}{2}$ and $\frac{3}{2} \in Q_1$

Hence, Q_1 is a group.

3. Que. If a, b be any two elements of a group (G, o) then prove that $(aob)^{-1} = b^{-1} o a^{-1}$.

Proof:- Here given (G, o) is a group let 'e' be identity element of G

Since $a, b \in G$

$\Rightarrow aob \in G$ [By closure axiom]

We know that by inverse axioms

if $a, b \in G$ and $ab = e = ba$

$\Rightarrow a^{-1} = b$ and $b^{-1} = a$

Take the expression

$(aob) o (b^{-1} o a^{-1})$

$= [a o (bob^{-1})] o a^{-1}$ [By Associative law]

$= [aoe] o a^{-1}$ [By inverse law, $aoa^{-1} = e$, $bob^{-1} = e$]

$= aoa^{-1}$ [By Identity law $aoe = a$]

$= e$ [By inverse axiom]

$\therefore (aob) o (b^{-1} o a^{-1}) = e$

$\Rightarrow$ Inverse of $aob = b^{-1} o a^{-1}$

$\Rightarrow (aob)^{-1} = b^{-1} o a^{-1}$; proved